DEEP-WATER SURFACE WAVES DUE TO ARBITRARY PERIODIC PRESSURES

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ABSTRACT

The deep-water wave problem of any initially prescribed periodic stationary surface pressure has been discussed. Asymptotic analyses of both the steady and the unsteady parts of the resulting wave integral are given. Exact solutions for the steady state waves have been obtained for the cases of concentrated and arbitrary symmetrical pressure distributions of finite extent. Some particular spatial pressure configurations over elliptic regions together with some characteristic features of the wave motion are discussed.

INTRODUCTION

The three-dimensional problem of deep-water surface waves due to a stationary periodic surface pressure has been investigated by Lamb (1904), Sretenskii (1956, 1957) and Wehausen and Laitone (1960). Of these, Lamb obtained the steady state integral for the wave elevation due to any symmetrical pressure and asymptotically evaluated this integral for the case of a concentrated pressure point. Investigation of Sretenskii was mainly concerned with the steady waves from a doubly modulated pressure distribution over a rectangular region, while Wehausen and Laitone gave only the velocity potential integrals for the general initial value problem with an arbitrary law of time-dependence and also for the steady state problem with harmonic pressures. In the second case, a first order asymptotic expression for the potential integral and the average rate of work done by the pressure were also given for symmetrical pressure distributions.

In this paper, the deep-water wave problem of any initially prescribed stationary periodic surface pressure has been solved by a method different from that used by Wehausen and Laitone. This method is equally applicable to the case of arbitrary time-dependence of the initial surface pressure. The following new points of interest are discussed. Firstly, methods of finding complete asymptotic expansions of both the steady and the unsteady parts of the resulting wave integral are given for different ranges of time and distance. The steady state wave integral has been exactly evaluated for concentrated and arbitrary symmetrical pressure distributions. An expression for the mean rate of work by an arbitrary pressure on the fluid is given. In all these cases,

particular spatial pressure configurations over elliptic regions, hitherto uninvestigated, are studied. Some characteristic features of both steady and unsteady waves are discussed.

Problem

A semi-infinite ocean of a heavy homogeneous inviscid liquid is initially at rest. Its undisturbed horizontal free surface forms the xy-plane while the z-axis is taken vertically upwards. Surface waves are excited and maintained in the ocean by the application of a surface pressure $p_0(x, y, t)$ of the form

$$p_0(x, y, t) = F(x, y)e^{i\omega t}, \quad t > 0, \qquad \ldots \quad (1)$$

in which the arbitrary function $F(x, y)$ is supposed to possess a double Fourier transform. If $\phi(x, y, z; t)$ be the velocity potential of the irrotational motion so generated, the equation of continuity is

$$\nabla^2\phi = 0, \quad z < 0, t > 0. \qquad \ldots \quad (2)$$

Introducing a frictional force equal to μ times the velocity, and assuming small oscillations in which squares of the velocities are negligible, we have, from Bernoulli's pressure equation,

$$\frac{p}{\rho} + gz - \mu\phi - \frac{\partial\phi}{\partial t} = F_1(t),$$

where p denotes the pressure at (x, y, z) at time t, ρ the uniform density of the liquid, g the acceleration due to gravity and $F_1(t)$ an arbitrary function of time t. On absorbing $F_1(t)$ in $\frac{\partial\phi}{\partial t}$, the surface condition (1) gives

$$g\rho\zeta = -F(x, y)e^{i\omega t} + \rho\left(\mu\phi + \frac{\partial\phi}{\partial t}\right)_{z=0}, \qquad \ldots \quad (3)$$

where ζ denotes the surface elevation at the point $(x, y, 0)$ at time t above the undisturbed surface.

The kinematical surface condition gives, to the above degree of approximation,

$$\frac{\partial\zeta}{\partial t} = -\left(\frac{\partial\phi}{\partial z}\right)_{z=0}, \quad t > 0. \qquad \ldots \quad (4)$$

Eliminating ζ between (3) and (4), we get

$$\frac{\partial^2\phi}{\partial t^2} + \mu\frac{\partial\phi}{\partial t} + g\frac{\partial\phi}{\partial z} = \frac{i\omega}{\rho}F(x, y)e^{i\omega t}, \; z = 0, \quad t > 0. \qquad \ldots \quad (5)$$

The conditions at infinity are

ϕ and its first and second derivatives $\to 0$

$$\text{as} \quad z \to -\infty, \quad t > 0. \qquad \ldots \quad (6)$$

The initial conditions are

$$\phi(x, y, 0; 0) = \phi_t(x, y, 0; 0) = 0. \quad .. \quad .. \quad .. \quad (7)$$

The problem is to find a solution $\phi(x, y, z; t)$ of (2) which satisfies the equations (2), (5), (7) together with the condition (6). The surface elevation ζ is then obtained by (3).

Solution

Assuming a double Fourier transform $\bar{\phi}(K_1, K_2, z; t)$ for the velocity potential function $\phi(x, y, z; t)$ namely

$$\bar{\phi}(K_1, K_2, z; t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\phi(x, y, z; t)e^{i(K_1x+K_2y)}dx\,dy, \quad .. \quad (8)$$

we obtain from (2) the differential equation

$$\frac{\partial^2\bar{\phi}}{\partial z^2} - (K_1^2+K_2^2)\bar{\phi} = 0.$$

Therefore

$$\bar{\phi} = A(K_1, K_2, t)e^{z(K_1^2+K_2^2)^{1/2}} \quad .. \quad .. \quad .. \quad (9)$$

in view of (6).

The result of transforming (5) is

$$\frac{\partial^2\bar{\phi}}{\partial t^2}+\mu\frac{\partial\bar{\phi}}{\partial t}+g\frac{\partial\bar{\phi}}{\partial z} = \frac{i\omega}{\rho}\bar{F}(K_1, K_2)e^{i\omega t},\ z = 0$$

in which $\bar{F}(K_1, K_2)$ is the double Fourier transform of $F(x, y)$. Using (9), we find that A satisfies the differential equation

$$\frac{\partial^2 A}{\partial t^2}+\mu\frac{\partial A}{\partial t}+\sigma^2 A = \frac{i\omega}{\rho}\bar{F}(K_1, K_2)e^{i\omega t} \quad .. \quad .. \quad .. \quad (10)$$

where
$$\sigma^2 = gK \text{ and } K = (K_1^2+K_2^2)^{\frac{1}{2}}. \quad .. \quad .. \quad .. \quad (11)$$

On putting $A = v(K_1, K_2, t)e^{-\frac{1}{2}\mu t}$ and neglecting μ^2, we obtain from (10) the equation

$$\frac{\partial^2 v}{\partial t^2}+\sigma^2 v = \frac{i\omega}{\rho}e^{(\frac{1}{2}\mu+i\omega)t}\bar{F}(K_1, K_2).$$

The real solution of this equation is

$$v = v_0(K_1, K_2)\cos\{\sigma t-\sigma_0(K_1, K_2)\}+\frac{i\omega}{\rho\sigma}\bar{F}(K_1, K_2)\int_0^t e^{(\frac{1}{2}\mu+i\omega)s}\sin\sigma(t-s)ds.$$

The initial conditions (7) make $v_0 = 0$.

Consequently,

$$A = \frac{i\omega}{\rho\sigma}\bar{F}(K_1, K_2)e^{i\omega t}\int_0^t e^{-(\frac{1}{2}\mu+i\omega)s}\sin\sigma s\,ds. \quad .. \quad .. \quad (12)$$

Inserting the last expression for A in (9), and inverting (8) by the Fourier double inversion theorem, we find that the velocity potential is

$$\phi(x, y, z; t) = \frac{i\omega}{2\pi\rho} e^{i\omega t} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{\sigma} \overline{F}(K_1, K_2) e^{Kz - i(K_1 x + K_2 y)} \int_0^t e^{-(\frac{1}{2}\mu + i\omega)s} \times \sin \sigma s \, ds \, dK_1 dK_2 \quad .. \quad (13)$$

If we put $\mu = 0$ in (13), the result agrees with that of Wehausen and Laitone (1960, 22.48) for the case of harmonic strengths of the initial pressure.

Equation (3) now gives, after a little simplification in which μ^2 has been neglected, the expression for the surface elevation as

$$g\rho\zeta = -F(x, y)e^{i\omega t} + \frac{i\omega}{2\pi} e^{-\frac{\mu t}{2}} \lim_{z \to 0-} \int\int_{-\infty}^{\infty} \frac{\overline{F}(K_1, K_2)\, e^{Kz - i(K_1 x + K_2 y)}}{\sigma^2 + i\mu\omega - \omega^2} \times \left[\sigma \sin \sigma t - \frac{i\mu\omega}{2\sigma} \sin \sigma t - (\mu + i\omega) \cos \sigma t\right] dK_1 dK_2 + \frac{i\mu\omega - \omega^2}{2\pi} e^{i\omega t} \int\int_{-\infty}^{\infty} \frac{\overline{F}(K_1, K_2)\, e^{-i(K_1 x + K_2 y)}}{\sigma^2 + i\mu\omega - \omega^2} dK_1 dK_2 . \quad .. \quad .. \quad (14)$$

The passage to the limit $z \to 0-$ in the second term of (14) is easily justified for finitely extended pressure distributions $F(x, y)e^{i\omega t}$. For a concentrated disturbance, this is justified by the results obtained.

Finally, the solutions of the present problem are to be obtained by passing to the limit $\mu \to 0+$ in (13) and (14).

We can write

$$\zeta = \zeta_1 + \zeta_2 , \quad .. \quad .. \quad .. \quad .. \quad (15)$$

where $g\rho\zeta_1$ and $g\rho\zeta_2$ are respectively the second and the sum of the first and the third terms on the r.h.s. of (14). Clearly, ζ_1 represents the transient part of ζ which dies out (even in the limit $\mu \to 0+$, as will be shown later) as $t \to \infty$, while ζ_2 represents the steady state oscillatory solution for the wave elevation ζ. The general solutions of the less general problems of (i) symmetrical and (ii) certain asymmetrical oscillatory pressure distributions over circular regions follow at once from (13) and (14).

(i) *Symmetrical pressure areas*

If $F(x, y) = f(r)$, $r = (x^2 + y^2)^{\frac{1}{2}}$, then

$$\overline{F}(K_1, K_2) = \bar{f}(K), \quad K = (K_1^2 + K_2^2)^{\frac{1}{2}},$$

where $\bar{f}(K) = \int_0^{\infty} \alpha f(\alpha) J_0(K\alpha)\, d\alpha$ is the zero-order Hankel transform of the function $f(r)$ (Bochner and Chandrasekharan 1949, Th. 40). By the same

theorem, the velocity potential $\phi(r, z; t)$ and the wave elevation $\zeta(r, t)$ in this case are found to be given by

$$\phi(r, z; t) = \frac{i\omega}{\rho} e^{i\omega t} \int_0^\infty \frac{K}{\sigma} \bar{f}(K) e^{Kz} J_0(Kr) \int_0^t e^{-(\frac{1}{2}\mu + i\omega)s} \sin \sigma s \, ds \, dK \quad .. \quad (16)$$

$$g\rho\zeta(r, t) = -f(r)e^{i\omega t} + i\omega e^{-\frac{\mu t}{2}} \lim_{z \to 0-} \int_0^\infty \frac{K e^{Kz} \bar{f}(K) J_0(Kr)}{\sigma^2 + i\mu\omega - \omega^2}$$

$$\times \left[\sigma \sin \sigma t - \frac{i\mu\omega}{2\sigma} \sin \sigma t - (\mu + i\omega) \cos \sigma t \right] dK$$

$$+ (i\mu\omega - \omega^2) e^{i\omega t} \int_0^\infty \frac{K \bar{f}(K) J_0(Kr)}{\sigma^2 + i\mu\omega - \omega^2} dK. \quad .. \quad (17)$$

Combining the first and the third terms on the r.h.s. of (17), we find the part of ζ in the steady state to be

$$\zeta_2 = -\frac{e^{i\omega t}}{\rho} \int_0^\infty \frac{K^2 \bar{f}(K) J_0(Kr)}{\sigma^2 + i\mu\omega - \omega^2} dK.$$

This is Lamb's well-known steady state integral for the wave elevation in this problem (i) (Lamb 1904, eqn. 123).

(ii) *Asymmetrical pressures on circular regions*

If $$F(x, y) = f_1(r) \cos m\theta + f_2(r) \sin m\theta,$$

where r, θ, z are cylindrical co-ordinates, and m is a positive integer, it can be easily shown that

$$\bar{F}(K_1, K_2) = e^{im\pi/2} [\bar{f}_1(K) \cos m\psi + \bar{f}_2(K) \sin m\psi]$$

where K, ψ are the plane polar co-ordinates in the (K_1, K_2)-plane, and $\bar{f}_1(K)$ and $\bar{f}_2(K)$ are the mth order Hankel transforms of the functions $f_1(r)$ and $f_2(r)$ respectively.

In this case, the velocity potential $\phi(r, \theta, z; t)$ and $\zeta(r, \theta; t)$ are found to be given by

$$\phi(r, \theta, z; t) = \frac{i\omega}{\rho} e^{i\omega t} \int_0^\infty \bar{F}_{12}(K, \theta) \frac{K}{\sigma} e^{Kz} J_m(Kr) \int_0^t e^{-(\frac{1}{2}\mu + i\omega)s} \sin \sigma s \, ds \, dK \quad (18)$$

$$g\rho\zeta(r, \theta; t) = -[f_1(r) \cos m\theta + f_2(r) \sin m\theta] e^{i\omega t}$$

$$+ i\omega e^{-\frac{\mu t}{2}} \lim_{z \to 0-} \int_0^\infty \frac{K \bar{F}_{12}(K, \theta) e^{Kz} J_m(Kr)}{\sigma^2 + i\mu\omega - \omega^2}$$

$$\times [\sigma \sin \sigma t - \frac{i\mu\omega}{2\sigma} \sin \sigma t - (\mu + i\omega) \cos \sigma t] \, dK$$

$$+ (i\mu\omega - \omega^2) e^{i\omega t} \int_0^\infty \frac{K \bar{F}_{12}(K, \theta) J_m(Kr)}{\sigma^2 + i\mu\omega - \omega^2} dK \quad .. \quad (19)$$

where $$\bar{F}_{12}(K, \theta) = \bar{f}_1(K) \cos m\theta + \bar{f}_2(K) \sin m\theta. \quad .. \quad .. \quad (20)$$

In (16) to (19), the final expressions are to be obtained by passing to the limit $\mu \to 0+$.

Asymptotic analysis of ζ_1.

A method of finding the complete asymptotic expansion of the transient part ζ_1 of ζ (in the limit $\mu \to 0+$) for a certain range of large values of r and t will now be given. We assume that the impressed pressure $p_0(x, y, t)$ is zero outside a finite region D of the xy-plane.

As defined in (15)

$$g\rho\zeta_1 = \frac{i\omega}{2\pi} e^{-\frac{\mu t}{2}} \lim_{z \to 0-} \int_{-\infty}^{\infty}\int \frac{\bar{F}(K_1, K_2) e^{Kz - i(K_1 x + K_2 y)}}{\sigma^2 + i\mu\omega - \omega^2}$$

$$\times \left[\sigma \sin \sigma t - \frac{i\mu\omega}{2\sigma} \sin \sigma t - (\mu + i\omega) \cos \sigma t\right] dK_1\, dK_2 \quad .. \quad .. \quad (21)$$

in which finally the limit $\mu \to 0+$ is to be taken.

Let

$$I = \lim_{z \to 0-} \int_{-\infty}^{\infty}\int \bar{F}(K_1, K_2) e^{Kz - i(K_1 x + K_2 y)} \cdot \frac{\sin \sigma t}{\sigma(\sigma^2 + i\mu\omega - \omega^2)} dK_1\, dK_2. \quad .. \quad (22)$$

Replacing $\bar{F}(K_1, K_2)$ by the corresponding double integral and then interchanging the orders of the two double integrations, we get

$$I = \int_D \int F(\alpha, \beta) d\alpha\, d\beta \,.\, I_1 \,. \quad .. \quad .. \quad .. \quad (23)$$

so that

$$gI_1 = \lim_{z \to 0-} \frac{1}{2\pi} \int_{-\infty}^{\infty}\int e^{Kz - i[K_1(x-\alpha) + K_2(y-\beta)]} \cdot \frac{\sin \sigma t}{\sigma(K - K')} dK_1\, dK_2$$

$$= \lim_{z \to 0-} \int_0^{\infty} e^{Kz} J_0(KR) \cdot \frac{K \sin \sigma t}{\sigma(K - K')} dK \quad .. \quad .. \quad .. \quad .. \quad (24)$$

where

$$\left.\begin{aligned} R^2 &= (x-\alpha)^2 + (y-\beta)^2 \\ K' &= \frac{\omega^2 - i\mu\omega}{g} \end{aligned}\right\} \quad .. \quad .. \quad .. \quad (25)$$

Replacing $J_0(KR)$ in (24) by its integral representation

$$J_0(KR) = \frac{2}{\pi} \int_0^{\frac{\pi}{2}} \cos(KR \sin \gamma)\, d\gamma$$

and then interchanging the orders of γ- and K-integrations, we get

$$\frac{\pi}{2} g I_1 = \int_0^{\frac{\pi}{2}} d\gamma \lim_{z \to 0-} \int_0^{\infty} K e^{Kz} \cos(KR \sin\gamma) \frac{\sin \sigma t}{\sigma(K-K')} dK.$$

Expanding $\sin \sigma t$ in powers of σt, and interchanging the orders of summation and K-integration, we obtain

$$\begin{aligned}
\frac{\pi}{2} g I_1 &= \int_0^{\frac{\pi}{2}} d\gamma \sum_{n=0}^{\infty} (-1)^n \frac{g^n t^{2n+1}}{(2n+1)!} \lim_{z \to 0-} \int_0^{\infty} \frac{K^{n+1}}{K-K'} e^{Kz} \cos(KR \sin\gamma)\, dK \\
&= \int_0^{\frac{\pi}{2}} d\gamma \sum_{n=0}^{\infty} (-1)^n \frac{g^n t^{2n+1}}{(2n+1)!} \sum_{r=0}^{n} K'^r \lim_{z \to 0-} \int_0^{\infty} K^{n-r} e^{Kz} \cos(KR \sin\gamma)\, dK \\
&\qquad + \frac{\pi}{2} \int_0^{\infty} \frac{J_0(KR)}{K-K'} dK. \quad \left(\frac{K'}{g}\right)^{\frac{1}{2}} \sin (gK't^2)^{\frac{1}{2}} \\
&= \int_0^{\frac{\pi}{2}} d\gamma \sum_{n=0}^{\infty} (-1)^n \frac{g^n t^{2n+1}}{(2n+1)!} \sum_{r=0}^{n} K'^r \frac{(n-r)!}{(R \sin\gamma)^{n-r+1}} \cos (n-r+1) \frac{\pi}{2} \\
&\qquad + \frac{\pi}{2} \left(\frac{K'}{g}\right)^{\frac{1}{2}} I_2 \sin (gK't^2)^{\frac{1}{2}}
\end{aligned}$$

where

$$I_2 = \int_0^{\infty} \frac{J_0(KR)}{K-K'} dK. \quad .. \quad .. \quad .. \quad .. \quad .. \quad .. \quad .. \quad .. \quad (26)$$

Since

$$\frac{1}{(2n+1)!} = \frac{B(2r+1,\, 2n-2r+1)}{\Gamma(2r+1)\Gamma(2n-2r+1)} = \frac{\int_0^1 s^{2n-2r}(1-s)^{2r} ds}{\Gamma(2r+1)\Gamma(2n-2r+1)},$$

we can apply Cauchy's rule for the multiplication of power series, and obtain

$$\frac{\pi}{2}\left[g I_1 - \left(\frac{K'}{g}\right)^{\frac{1}{2}} I_2 \sin \tau_1'\right] = 4 \left(\frac{K'}{g}\right)^{\frac{1}{2}} \frac{\tau_1}{\tau_1'} \int_0^{\frac{\pi}{2}} \operatorname{cosec} \gamma\, d\gamma \int_0^1 \cos \tau_1' (1-s) L(2\tau_2 s^2)\, ds \quad (27)$$

where we define the functions L and M by

$$\left.\begin{aligned}
L(x) &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1} 2^{2n} (2n)!}{(4n+1)!} \\
M(x) &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+2} 2^{2n+1} (2n+1)!}{(4n+3)!}
\end{aligned}\right\} \quad .. \quad .. \quad (28)$$

and some dimensionless variables τ etc. by

$$\left.\begin{aligned}
&\tau = \frac{gt^2}{4r}, \quad \tau_1 = \frac{gt^2}{4R}, \quad \tau_2 = \tau_1 \operatorname{cosec} \gamma \\
&\tau' = \omega t, \quad \tau_1' = (gK't^2)^{\frac{1}{2}}, \quad r' = \frac{r}{r_0}
\end{aligned}\right\} \quad .. \quad .. \quad (29)$$

where $r_0^2 = (\alpha^2+\beta^2)_{\max.}$, (α, β) being a point in the domain of $F(x, y)$.
It is known that

$$L(x) = (\pi x)^{\frac{1}{2}}\left[\cos\frac{x}{2}\,C\left(\frac{x}{2}\right)+\sin\frac{x}{2}\,S\left(\frac{x}{2}\right)\right] \quad .. \quad .. \quad .. \quad .. \quad .. \quad (30.1)$$

$$= (\pi x)^{\frac{1}{2}}\left[\frac{1}{2}\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)-\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\int_0^\infty e^{-\lambda\sqrt{2x}}\sin\lambda^2 d\lambda\right],\ x \geqslant 0 \quad .. \quad (30.2)$$

and

$$M(x) = (\pi x)^{\frac{1}{2}}\left[\sin\frac{x}{2}\,C\left(\frac{x}{2}\right)-\cos\frac{x}{2}\,S\left(\frac{x}{2}\right)\right], \quad .. \quad .. \quad .. \quad .. \quad .. \quad (30.3)$$

(Ryshik and Gradstein 1957, 6·233 (2, 3), 6·236 (3)) where $C(x)$ and $S(x)$ denote Fresnel's integrals defined by

$$[C(x),\ S(x)] = \frac{1}{(2\pi)^{\frac{1}{2}}}\int_0^x \frac{[\cos t,\ \sin t]}{t^{\frac{1}{2}}}\,dt.$$

By (30.2),

$$\text{the } s\text{-integral of (27)} = \left(\frac{\pi\tau_2}{2}\right)^{\frac{1}{2}}\int_0^1 s\cos\tau_1'(1-s)\left[\cos(\tau_2 s^2)+\sin(\tau_2 s^2)\right]ds$$

$$-2\tau_2^{\frac{1}{2}}\int_0^\infty \sin\lambda^2 d\lambda\int_0^1 s\cos\tau_1'(1-s)e^{-2\lambda s\tau_2^{\frac{1}{2}}}\,ds \quad .. \quad (31)$$

in which the orders of s- and λ-integrals have been interchanged in the second term.

In the asymptotics that follow, the range of large values of r and t is defined by the conditions:

$$r' \gg 1, \quad r'^2 \gg \tau \gg \tau' \gg 1. \quad .. \quad .. \quad .. \quad (32)$$

Applying Erdélyi's theorem (Erdélyi 1955, 2.9) to the first term of (31) in which τ_2 is predominantly large, we have now the first term of (31)

$$= -\left(\frac{\pi\tau_2}{2}\right)^{\frac{1}{2}}\sum_{n=0}^{\infty}\frac{l^{(n)}(0)}{\tau_2^{n+1}}\left[\cos\left\{\tau_2-(n+1)\frac{\pi}{2}\right\}+\sin\left\{\tau_2-(n+1)\frac{\pi}{2}\right\}\right]$$

$$+\left(\frac{\pi}{8\tau_2}\right)^{\frac{1}{2}}\left[L\left(\frac{\tau_1'^2}{2\tau_2}\right)+1-M\left(\frac{\tau_1'^2}{2\tau_2}\right)\right]\cos\tau_1'$$

$$+\frac{\pi\tau_1'}{4\tau_2}\sin\tau_1'\sin\frac{\tau_1'^2}{4\tau_2} \quad .. \quad .. \quad .. \quad .. \quad .. \quad .. \quad .. \quad (33)$$

where

$$l(v) = -\tfrac{1}{2}\cos\{\tau_1'(1-\sqrt{1-v})\} \quad .. \quad .. \quad .. \quad (34)$$

Next, evaluating the s-integral of the second term of (31), we get the second term of (31)

$$= 2\tau_2^{\frac{1}{2}} \int_0^\infty e^{-2\lambda\tau_2^{\frac{1}{2}}} \sin\lambda^2 \cdot \frac{(2\lambda\tau_2^{\frac{1}{2}}+1)(4\tau_2\lambda^2+\tau_1'^2)-2\tau_1'^2}{(4\tau_2\lambda^2+\tau_1'^2)^2}\, d\lambda$$

$$-2\tau_2^{\frac{1}{2}}[(I_3-2\tau_1'^2 I_4)\cos\tau_1' + 4\tau_2^{\frac{1}{2}}\tau_1' I_5 \sin\tau_1'] \quad .. \quad .. \quad .. \quad (35)$$

where

$$I_3, I_4, I_5 = \int_0^\infty \frac{(1, 1, \lambda)\sin\lambda^2}{(4\tau_2\lambda^2+\tau_1'^2)^{1,2,2}}\, d\lambda, \text{ taken in order.}$$

Since $\tau_2 \gg 1$, the asymptotic expansion of the first term of (35) can be completely found out by Watson's Lemma (Watson 1944, § 8.3). Using some results on Stieltjes' transforms (Erdélyi 1954, 14.2 (31, 33, 48) and certain properties of sine and cosine integrals, we can write

$$I_3 = \frac{1}{\tau_1'^2}\left(\frac{\pi}{8}\right)^{\frac{1}{2}}\left[M\left(\frac{\tau_1'^2}{2\tau_2}\right)+L\left(\frac{\tau_1'^2}{2\tau_2}\right)-\left(\frac{\pi}{2\tau_2}\right)^{\frac{1}{2}}\tau_1' \sin\frac{\tau_1'^2}{4\tau_2}\right],$$

$$I_5 = \frac{1}{32\,\tau_2^2}\left[\int_0^\infty \frac{\cos(KR\sin\gamma)}{K-K'}\, dK + i\pi\cos\frac{\tau_1'^2}{4\tau_2}+\pi\sin\frac{\tau_1'^2}{4\tau_2}\right]$$

while I_4 is easily evaluated from I_3.

Taking all these results for (35) together with (33), we find the s-integral of (27)

$$= -\left(\frac{\pi\tau_2}{2}\right)^{\frac{1}{2}} \sum_{n=0}^{\infty} \frac{l^{(n)}(0)}{\tau_2^{n+1}}\left[\cos\left\{\tau_2-(n+1)\frac{\pi}{2}\right\}+\sin\left\{\tau_2-(n+1)\frac{\pi}{2}\right\}\right]$$

$$+\left(\frac{1}{\tau_1'^2\tau_2}+\ldots\right)-\frac{1}{4}\frac{\tau_1'}{\tau_2}\sin\tau_1'\int_0^\infty \frac{\cos(KR\sin\gamma)}{K-K'}\, dK+\frac{\pi}{4}\frac{\tau_1'}{\tau_2}e^{-i\tau_1'}\cos\frac{\tau_1'^2}{4\tau_2}. \quad .. \quad .. \quad (36)$$

It is now clear from (27) that the complete asymptotic expansion of I_1 (and hence of I and ζ_1) can be found out, the γ-integrals involved being calculated as shown below. However, retaining only up to the second order terms, we get, from (27),

$$\frac{\pi}{2}\left[gI_1-\pi\left(\frac{K'}{g}\right)^{\frac{1}{2}}e^{-i\tau_1'}J_0(K'R)\right] = \left(\frac{2\pi K'}{g}\right)^{\frac{1}{2}}\frac{\tau_1^{\frac{1}{2}}}{\tau_1'}\int_0^{\pi/2}\left(\sin\frac{\tau_1}{\sin\gamma}-\cos\frac{\tau_1}{\sin\gamma}\right)\frac{d\gamma}{\sin^{\frac{1}{2}}\gamma}$$

$$+3\text{rd order terms}$$

$$= \pi\left(\frac{K'}{g}\right)^{\frac{1}{2}}\frac{\tau_1}{\tau_1'}\int_0^\infty \{J_{\frac{1}{2}}(\tau_1\cosh\lambda)-J_{-\frac{1}{2}}(\tau_1\cosh\lambda)\}\, d\lambda+3\text{rd order terms}$$

$$= -\frac{\pi^2}{2}\left(\frac{K'}{g}\right)^{\frac{1}{2}}\frac{\tau_1}{\tau_1'}\left\{J_{\frac{1}{2}}\left(\frac{\tau_1}{2}\right)Y_{\frac{1}{2}}\left(\frac{\tau_1}{2}\right)-J_{-\frac{1}{2}}\left(\frac{\tau_1}{2}\right)Y_{-\frac{1}{2}}\left(\frac{\tau_1}{2}\right)\right\}+3\text{rd order terms}$$

(Erdélyi 1953, 7.14.2 (65)).

Since τ_1 is large, we may use the asymptotic series for the Bessel and Neumann functions written above. Then

$$\frac{\pi}{2}\left[gI_1-\pi\left(\frac{K'}{g}\right)^{\frac{1}{2}}e^{-i\tau_1'}J_0(K'R)\right]=\frac{\pi}{\tau_1'}\left(\frac{2K'}{g}\right)^{\frac{1}{2}}\left(\sin\tau_1-\frac{3}{8\tau_1}\cos\tau_1\right)+\text{3rd order terms.}$$

Expanding R^{-1} in τ_1 in powers of r^{-1} and retaining only those terms which contribute up to the second order terms, we have, from (23),

$$I=\frac{2^{\frac{1}{2}}\pi}{tg^2}\,\mathfrak{I}_m\left(We^{\frac{igt^2}{4r}}\right)+\pi\left(\frac{K'}{g^3}\right)^{\frac{1}{2}}e^{-it\sqrt{gK'}}\int_D\int F(\alpha,\beta)J_0(K'R)\,d\alpha\,d\beta$$
$$+\text{3rd order terms} \quad .. \quad .. \quad .. \quad .. \quad .. \quad .. \quad .. \quad (37)$$

where $\mathfrak{I}_m(z) \equiv$ the imaginary part of z,

$$\left.\begin{aligned} W &= \bar{F}-i\frac{gt^2}{8r^3}\left(3\bar{F}_{ww}-\nabla_w^2\bar{F}\right)-\frac{3ir}{2gt^2}\bar{F},\\ \bar{F} &\equiv \bar{F}(w\cos\theta,\, w\sin\theta),\ w=\frac{gt^2}{4r^2}\\ \bar{F}_{ww} &\equiv \frac{\partial^2\bar{F}}{\partial w^2},\ \nabla_w^2\equiv\frac{\partial^2}{\partial w^2}+\frac{1}{w}\frac{\partial}{\partial w}+\frac{1}{w^2}\cdot\frac{\partial^2}{\partial\theta^2}. \end{aligned}\right\} \quad .. \quad .. \quad (38)$$

The other members of (21) being proportional to $\frac{\partial I}{\partial t}$, $\frac{\partial^2 I}{\partial t^2}$, we easily find

$$g\rho\zeta_1=\frac{\omega}{2^{\frac{1}{2}}gr}\left[\frac{igt}{r}\,\mathfrak{I}_m\left(W_2e^{\frac{igt^2}{4r}}\right)+2\omega\,\mathfrak{R}_e\left(W_1e^{\frac{igt^2}{4r}}\right)\right]+\text{3rd order terms} \quad (39)$$

under the restrictions (32),
where

$$W_1=W+\frac{2ir}{gt^2}\bar{F}-\frac{i}{r}\bar{F}_w,\ W_2=W_1-\frac{i}{r}\bar{F}_w, \quad .. \quad .. \quad (40)$$

W being given by (38).
The lowest order term is

$$g\rho\zeta_1\simeq\frac{\omega}{2^{\frac{1}{2}}gr}\left[2\omega\mathfrak{R}_e\left(\bar{F}e^{\frac{igt^2}{4r}}\right)+\frac{igt}{r}\,\mathfrak{I}_m\left(\bar{F}e^{\frac{igt^2}{4r}}\right)\right] \quad .. \quad .. \quad (41)$$

under the restrictions (32).
When $\qquad F(x,y)=f(r),\ \bar{F}(w\cos\theta,\, w\sin\theta)=\bar{f}(w),$
where $\bar{f}(k)$ is the zero-order Hankel transform of $f(r)$; then

$$\left.\begin{aligned} W_1 &= \bar{f}(w)-i\left\{\frac{gt^2}{4r^3}\bar{f}''(w)+\frac{1}{2r}\bar{f}'(w)-\frac{r}{2gt^2}\bar{f}(w)\right.\\ W_2 &= W_1-\frac{i}{r}\bar{f}'(w) \end{aligned}\right\} \quad .. \quad .. \quad (42)$$

and ζ_1 is obtained from (39).

When

$$F(x, y) = f_1(r) \cos m\theta + f_2(r) \sin m\theta,$$

$$\bar{F}(w \cos \theta, w \sin \theta) = e^{im\pi/2} [\bar{f}_1(w) \cos m\theta + \bar{f}_2(w) \sin m\theta],$$

$\bar{f}_1(K)$ and $\bar{f}_2(K)$ being the mth order Hankel transforms of $f_1(r)$ and $f_2(r)$ respectively. The corresponding expression for ζ_1 follows from (39) by (38) and (40).

Decay of the unsteady state

By (24)

$$\tfrac{1}{2} g^{\frac{1}{2}} I_1 = \lim_{z \to 0-} \int_0^\infty G(k) \sin (gK^2t^2)^{\frac{1}{2}} \, dK$$

where

$$G(K) = \frac{K^2}{K^2 - K'} J_0(K^2R) e^{K^2 z}.$$

The K-integral is a Fourier integral and the function $G(K)$ satisfies all the conditions required for this integral to have an asymptotic expansion as $t \to \infty$, r and ω remaining fixed.
Therefore, we have (Erdélyi 1955, 2.8)

$$\tfrac{1}{2} g^{\frac{1}{2}} I_1 = \lim_{z \to 0-} \sum_{n=1}^{\infty} (-1)^n \frac{G^{(2n)}(0)}{(g^{\frac{1}{2}}t)^{2n+1}}.$$

Consequently, from (21),

$$g\rho\zeta_1 = \frac{i\omega}{2\pi} e^{-\frac{\mu t}{2}} \sum_{n=1}^{\infty} \frac{a_n}{t^{2n+3}} \left[-(2n+1)(2n+2) + (\mu + i\omega)(2n+1)t - \frac{i\mu\omega}{2} t^2 \right]$$

$$\to \frac{i\omega}{2\pi} \sum_{n=1}^{\infty} \frac{(a_n)_{\mu=0+}}{t^{2n+3}} \cdot (2n+1)[i\omega t - (2n+2)], \text{ as } \mu \to 0+, \quad .. \quad .. \quad (43)$$

where

$$a_n = (-1)^n 2 g^{-(n+2)} \int_D \int F(\alpha, \beta) \left\{ \lim_{z \to 0-} G^{(2n)}(0) \right\} d\alpha \, d\beta. \quad .. \quad (44)$$

To find $\lim_{z \to 0-} G^{(2n)}(0)$, we note that

$$G^{(2n)}(0) = 2n(2n-1) \left[(2K)^{2n-2} G_1^{(2n-2)}(K^2) + \frac{(2n-2)(2n-3)}{1!} (2K)^{2n-4} G_1^{(2n-3)}(K^2) \right.$$

$$+ \frac{(2n-2)(2n-3)(2n-4)(2n-5)}{2!} (2K)^{2n-6} G_1^{(2n-4)}(K^2)$$

$$\left. + \ldots \right]_{K=0}$$

(Ryshik and Gradstein 1957, 0·432)

where

$$G_1(K^2) = \frac{e^{K^2 z} J_0(K^2 R)}{K^2 - K'}$$

and so

$$G_1^{(n)}(K^2) = \sum_{r=0}^{n} (-1)^r \binom{n}{r} r!\,(K^2-K')^{-(r+1)} \frac{d^{n-r}}{d(K^2)^{n-r}} \left[e^{K^2 z} J_0(K^2 R) \right]$$

$$\to \sum_{r=0}^{n} (-1)^r \binom{n}{r} r!\,(K^2-K')^{-(r+1)} R^{n-r} J_0^{(n-r)}(K^2 R) \text{ as } z \to 0- .$$

Consequently,

$$\lim_{z \to 0-} G^{(2n)}(0) = -\frac{(2n)!}{(n-1)!} \sum_{r=0}^{n-1} \binom{n-1}{r} r! \left(\frac{g}{\omega^2}\right)^{r+1} R^{n-r-1} \lim_{K \to 0+} J_0^{(n-r-1)}(K^2 R), \quad (45)$$

on making $\mu \to 0+$.

From (43), we find that ζ_1 dies out like t^{-4} (or t^{-5}) as $t \to \infty$. We also note that when μ is not an infinitesimal, the law of decay of ζ_1 with time is an exponential one. The use of μ thus physically signifies a quicker attainment of the steady state in presence of frictional forces.

Analysis of ζ_2 or the steady state

Lemma.

$$\int_0^\infty \frac{J_0(Kq)}{K-K'} dK = -\frac{\pi}{2} [H_0(K'q) + Y_0(K'q) + 2iJ_0(K'q)], \; q > 0, \quad .. \quad (46)$$

where $H_0(x)$ is Struve's function of order zero.
For

$$\int_0^\infty \frac{J_0(Kq)}{K-K'} dK = \frac{\pi}{2} [H_0(-K'q) - Y_0(-K'q)], \; q > 0$$

(Erdélyi 1954, 14.3 (3)).

Since

$$H_0(ze^{im\pi}) = e^{im\pi} H_0(z)$$

$$Y_0(ze^{im\pi}) = Y_0(z) + 2im J_0(z), \; m = 1, 2, 3, \ldots$$

the result (46) follows.

(*a*) *Concentrated pressure*

(i) *Exact solution*

If

$$p_0(x, y, t) = \frac{P}{2\pi r} \delta(r) e^{i\omega t},$$

the wave-elevation ζ_2 in the steady state is given by (17)

$$g\rho\zeta_2 = -\frac{P}{2\pi r}\delta(r)e^{i\omega t} - \frac{K'P}{2\pi}e^{i\omega t}\int_0^\infty \frac{KJ_0(Kr)}{K-K'}\,dK$$

$$= -\frac{P}{2\pi r}\delta(r)e^{i\omega t} - \frac{K'P}{2\pi r}e^{i\omega t} + \frac{K'^2P}{4}e^{i\omega t}[H_0(K'r) + Y_0(K'r) + 2iJ_0(K'r)]$$

by (46).

On putting $\mu = 0$, we have, for $r > 0$,

$$g\rho\zeta_2 = -\frac{\omega^2 P}{2\pi g r}e^{i\omega t} + \frac{P\omega^4}{4g^2}e^{i\omega t}\left[H_0\left(\frac{\omega^2 r}{g}\right) + Y_0\left(\frac{\omega^2 r}{g}\right) + 2iJ_0\left(\frac{\omega^2 r}{g}\right)\right] \quad .. \quad (47)$$

which gives the exact value of ζ_2 for concentrated pressure.

(ii) *Asymptotic solution*

For $\frac{\omega^2 r}{g} \gg 1$, the complete asymptotic development of ζ_2 can be at once obtained from (47). Retaining, however, only up to the second order terms, we have

$$g\rho\zeta_2 = \frac{P\omega^3}{(2\pi r g^3)^{\frac{1}{2}}}\left(i - \frac{g}{8\omega^2 r}\right)e^{-i\left(\frac{\omega^2 r}{g} - \omega t - \frac{\pi}{4}\right)} + O\left[\left(\frac{\omega^2 r}{g}\right)^{-\frac{5}{2}}\right]. \quad .. \quad (48)$$

The first order term of (48) agrees with that obtained by Lamb (1904, eqn. (135)) for this case.

(*b*) *Finitely extended symmetrical pressure. Exact solution*

If $$p_0(x, y, t) = f(r)e^{i\omega t},$$

where $f(r) = 0$ when $r > a$,

the wave-elevation ζ_2 in the steady state is given by (17)

$$g\rho\zeta_2 = -K'e^{i\omega t}\int_0^\infty \frac{K\bar{f}(K)J_0(Kr)}{K-K'}\,dK, \; r > a$$

where

$$\bar{f}(K) = \int_0^a \alpha f(\alpha)J_0(K\alpha)d\alpha.$$

On interchanging the orders of α- and K-integrations, we get

$$g\rho\zeta_2 = -K'e^{i\omega t}\int_0^a \alpha f(\alpha)d\alpha \int_0^\infty \frac{K}{K-K'}J_0(K\alpha)J_0(Kr)dK, \; r > a. \quad .. \quad (49)$$

Neumann's addition theorem for Bessel functions gives for $r > \alpha$,

$$J_0(K\alpha)J_0(Kr) = \frac{1}{\pi}\int_0^\pi J_0(K\lambda)d\theta \quad .. \quad .. \quad .. \; (50\text{A})$$

$$J_0(K\alpha)Y_0(Kr) = \frac{1}{\pi}\int_0^\pi Y_0(K\lambda)d\theta \quad .. \quad .. \quad .. \; (50\text{B})$$

where

$$\lambda^2 = r^2+\alpha^2-2r\alpha\cos\theta. \quad .. \quad .. \quad .. \quad .. \quad (51)$$

Hence, after a change of orders of integrations in (49) by (50A), we get

$$\text{the } K\text{-integral of (49)} = \frac{1}{\pi}\int_0^\pi \frac{d\theta}{\lambda} - \frac{K'}{2}\int_0^\pi [H_0(K'\lambda)+Y_0(K'\lambda)+2iJ_0(K'\lambda)]d\theta, \quad (52)$$

on using (46) and the well-known result

$$\int_0^\infty J_0(K\lambda)dK = \frac{1}{\lambda}, \quad \lambda > 0.$$

In (52), expanding $H_0(K'\lambda)$ in a series of powers of $K'\lambda$ and integrating term by term, we obtain

$$\int_0^\pi H_0(K'\lambda)d\theta = \sum_{n=0}^\infty (-1)^n \left[\frac{K'^2}{4}(r^2-\alpha^2)\right]^{n+\frac{1}{2}} [\Gamma(n+\tfrac{3}{2})]^{-2} \times \int_0^\pi \left(\frac{r^2+\alpha^2}{r^2-\alpha^2} - \frac{2r\alpha}{r^2-\alpha^2}\cos\theta\right)^{n+\frac{1}{2}} d\theta$$

$$= \pi \sum_{n=0}^\infty (-1)^n \left[\frac{K'^2}{4}(r^2-\alpha^2)\right]^{n+\frac{1}{2}} [\Gamma(n+\tfrac{3}{2})]^{-2} P_{n+\frac{1}{2}}\left(\frac{r^2+\alpha^2}{r^2-\alpha^2}\right), \quad r > \alpha, \quad .. \quad (53)$$

in which we have used Laplace's integral for the Legendre function $P_\nu(z)$. This integral also gives

$$\int_0^\pi \frac{d\theta}{\lambda} = \frac{\pi}{(r^2-\alpha^2)^{\frac{1}{2}}} P_{-\frac{1}{2}}\left(\frac{r^2+\alpha^2}{r^2-\alpha^2}\right), \quad r > \alpha. \quad .. \quad .. \quad (54)$$

Expressing the Legendre functions in (53) and (54) in terms of hypergeometric series in powers of $\frac{\alpha}{r}$, and evaluating the remaining integrals of (52) by means of (50A and 50B), we find

$$\text{the } K\text{-integral of (49)} = \frac{1}{r}\,{}_2F_1\left(\tfrac{1}{2}, \tfrac{1}{2}; 1, \frac{\alpha^2}{r^2}\right) - \frac{\pi K'}{2}\sum_{n=0}^\infty (-1)^n \frac{(K'r/2)^{2n+1}}{[\Gamma(n+\frac{3}{2})]^2}\cdot {}_2F_1\left(-n-\tfrac{1}{2}, -n-\tfrac{1}{2}; 1, \frac{\alpha^2}{r^2}\right) - \frac{\pi K'}{2} J_0(K'\alpha)[Y_0(K'r)+2iJ_0(K'r)], \quad r > \alpha.$$

Letting $\mu \to 0+$, we get from (49)

$$g\rho\zeta_2 e^{-i\omega t} = -\frac{\pi}{2}\left(\frac{\omega^2}{g}\right)^2 \sum_{n=0}^\infty (-1)^n \frac{(\omega^2 r/2g)^{2n-1}}{[\Gamma(n+\frac{1}{2})]^2}\cdot \int_0^a \alpha f(\alpha)\,{}_2F_1\left(-n+\tfrac{1}{2}, -n+\tfrac{1}{2}; 1, \frac{\alpha^2}{r^2}\right) d\alpha + \frac{\pi}{2}\left(\frac{\omega^2}{g}\right)^2 \left[Y_0\left(\frac{\omega^2 r}{g}\right)+2iJ_0\left(\frac{\omega^2 r}{g}\right)\right] \bar{f}\left(\frac{\omega^2}{g}\right), \quad r > a. \quad .. \quad .. \quad .. \quad (55)$$

This gives the exact solution for ζ_2 in the present case.

(c) *Asymptotics for finitely extended pressure area*

If the impressed pressure $p_0(x, y, t)$ vanishes outside a finite region D of the xy-plane, we have, from (14),

$$g\rho\zeta_2 = -\frac{K'}{2\pi}e^{i\omega t}\int_{-\infty}^{\infty}\int\frac{\bar{F}(K_1, K_2)}{K-K'}e^{-i(K_1x+K_2y)}dK_1dK_2,\ x, y \text{ outside } D.$$

On changing the order of the (K_1, K_2)-integral with that of the integral for $\bar{F}$, we get

$$g\rho\zeta_2 = -\frac{K'}{2\pi}e^{i\omega t}\int_D\int F(\alpha, \beta)d\alpha\, d\beta\,.\frac{1}{2\pi}\int_{-\infty}^{\infty}\int\frac{e^{-i[K_1(x-\alpha)+K_2(y-\beta)]}}{K-K'}dK_1\,dK_2$$

$$= -\frac{K'}{2\pi}e^{i\omega t}\int_D\int F(\alpha, \beta)\,d\alpha\,d\beta\int_0^{\infty}\frac{KJ_0(KR)}{K-K'}\,dK \quad .. \quad .. \quad .. \quad (56)$$

where

$$R^2 = (x-\alpha)^2+(y-\beta)^2.$$

As in (*b*) above,

$$\text{the } K\text{-integral of (56)} = \frac{1}{R}-\frac{\pi K'}{2}\,[H_0(K'R)+Y_0(K'R)+2iJ_0(K'R)]. \quad .. \quad (57)$$

In the asymptotic expansions that follow, we assume

$$\frac{r}{r_0}\gg 1,\quad \frac{\omega^2 r}{g}\gg 1,\quad \left(\frac{r}{r_0}\right)^2\gg\frac{\omega^2 r}{g}, \quad .. \quad .. \quad .. \quad (58)$$

where

$$r_0^2 = (\alpha^2+\beta^2)_{\max.},\ r^2 = x^2+y^2.$$

Using now the asymptotic series for the different functions in the second term on the r.h.s. of (57), we obtain

$$g\rho\zeta_2 = \frac{\omega^3}{(2\pi g^3)^{\frac{1}{2}}}e^{i\left(\omega t+\frac{\pi}{4}\right)}\int\int_D\frac{1}{R^{\frac{1}{2}}}\left(i-\frac{g}{8\omega^2R}\right)e^{-\frac{i\omega^2}{g}R}F(\alpha, \beta)d\alpha d\beta+O\left[\left(\frac{\omega^2 r}{g}\right)^{-\frac{1}{2}}\right].$$

Expanding $R^{-\frac{1}{2}}$, R, $R^{-\frac{3}{2}}$ appropriately, we have, to the above degree of approximation,

$$g\rho\zeta_2 = \frac{i\omega^3}{(2\pi r g^3)^{\frac{1}{2}}}e^{-i\left(\frac{\omega^2 r}{g}-\omega t-\frac{\pi}{4}\right)}\int\int_D\left[1+\frac{1}{2r}\left\{(\alpha\cos\theta+\beta\sin\theta)\right.\right.$$

$$\left.\left.-\frac{i\omega^2}{g}(-\alpha\sin\theta+\beta\cos\theta)^2+\frac{ig}{4\omega^2}\right\}\right]F(\alpha, \beta)e^{\frac{i\omega^2}{g}(\alpha\cos\theta+\beta\sin\theta)}\,d\alpha d\beta$$

$$+O\left[\left(\frac{\omega^2 r}{g}\right)^{-\frac{1}{2}}\right]$$

$$= -\left(\frac{2\pi}{rg^3}\right)^{\frac{1}{2}}\omega^3 e^{-i\left(\frac{\omega^2 r}{g}-\omega t+\frac{\pi}{4}\right)}\left[\bar{F}+\frac{ig}{8\omega^2 r}(4\bar{F}_{\theta\theta}+\bar{F})\right]+O\left[\left(\frac{\omega^2 r}{g}\right)^{-\frac{1}{2}}\right] \quad (59)$$

under the restrictions (58),

where

$$\bar{F} \equiv \bar{F}\left(\frac{\omega^2}{g}\cos\theta, \frac{\omega^2}{g}\sin\theta\right), \quad \bar{F}_{\theta\theta} = \frac{\partial^2 \bar{F}}{\partial\theta^2}.$$

When

$$F(x, y) = f(r)$$

and D is a circle of radius a,

$$\bar{F}(K_1, K_2) = \bar{f}(K)$$

where

$$\bar{f}(K) = \int_0^a \alpha f(\alpha) J_0(K\alpha)\, d\alpha, \quad K = (K_1^2 + K_2^2)^{\frac{1}{2}}.$$

Therefore, the above expression in this case reduces to

$$g\rho\zeta_2 = -\left(\frac{2\pi}{rg^3}\right)^{\frac{1}{2}} \omega^3 e^{-i\left(\frac{\omega^2 r}{g} - \omega t + \frac{\pi}{4}\right)} \left(1 + \frac{ig}{8\omega^2 r}\right) \bar{f}\left(\frac{\omega^2}{g}\right) + O\left[\left(\frac{\omega^2 r}{g}\right)^{-\frac{3}{2}}\right] \quad \ldots \quad (60)$$

When

$$F(x, y) = f_1(r)\cos m\theta + f_2(r)\sin m\theta,$$

and D is a circle of radius a,

$$\bar{F}(K_1, K_2) = e^{im\pi/2}\left[\bar{f}_1(K)\cos m\psi + \bar{f}_2(K)\sin m\psi\right]$$

where

$$K = (K_1^2 + K_2^2)^{\frac{1}{2}}, \quad \psi = \tan^{-1}\frac{K_2}{K_1}.$$

In this case, equation (59) gives

$$\begin{aligned} g\rho\zeta_2 = {} & -\omega^3\left(\frac{2\pi}{rg^3}\right)^{\frac{1}{2}} e^{-i\left(\frac{\omega^2 r}{g} - \omega t + \frac{\pi}{4} - \frac{m\pi}{2}\right)} \\ & \times \left[\bar{f}_1\left(\frac{\omega^2}{g}\right)\cos m\theta + \bar{f}_2\left(\frac{\omega^2}{g}\right)\sin m\theta\right] \cdot \left[1 + \frac{ig}{8\omega^2 r}(1 - 4m^2)\right] \\ & + O\left[\left(\frac{\omega^2 r}{g}\right)^{-5/2}\right]. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (61) \end{aligned}$$

(*d*) The mean rate at which an impressed pressure

$$p_0(x, y, t) = F(x, y)\begin{matrix}\cos \\ \sin\end{matrix}(\omega t) \quad \ldots \quad \ldots \quad \ldots \quad (62)$$

does work in generating waves can easily be obtained.

From (14), we have

$$g\rho\zeta_2 = -F(x, y)e^{i\omega t} - \frac{K'}{2\pi}e^{i\omega t}\int\int_{-\infty}^{\infty} F(\alpha, \beta)\, d\alpha\, d\beta \cdot$$

$$\times \left[\frac{1}{R} - \frac{\pi K'}{2}\{H_0(K'R) + Y_0(K'R) + 2iJ_0(K'R)\}\right]$$

as in (57) above. Putting $\mu = 0$, we get

$$g\rho\zeta_2 = -[F(x,y)+G(x,y)]\,e^{i\omega t}+\frac{i}{2}\frac{\omega^4}{g^2}e^{i\omega t}\int\int_{-\infty}^{\infty}F(\alpha,\beta)J_0\left(\frac{\omega^2R}{g}\right)d\alpha d\beta \quad .. \quad (63)$$

where

$$G(x,y)=\frac{\omega^2}{2\pi g}\int\int_{-\infty}^{x}F(\alpha,\beta)\left[\frac{1}{R}-\frac{\pi\omega^2}{2g}\left\{H_0\left(\frac{\omega^2R}{g}\right)+Y_0\left(\frac{\omega^2R}{g}\right)\right\}\right]d\alpha d\beta$$

$$= \text{a real expression.}$$

Taking the real or imaginary parts of $\frac{\partial\zeta_2}{\partial t}$ according as p_0/F in (62) is $\cos\omega t$ or $\sin\omega t$, we find, in either case, that the mean rate of work is

$$W_{\text{av.}}=\frac{\omega^5}{4\rho g^3}\int\int_{-\infty}^{\infty}F(x,y)\cdot\left(\int\int_{-\infty}^{\infty}F(\alpha,\beta)J_0\left(\frac{\omega^2R}{g}\right)d\alpha d\beta\right)dxdy.$$

Since

$$J_0\left(\frac{\omega^2R}{g}\right)=\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{\frac{i\omega^2R}{g}\cos(\psi-\theta)}\,d\psi, \quad \theta=\tan^{-1}\frac{y-\beta}{x-\alpha},$$

it follows after a change of order of ψ-integration,

$$W_{\text{av.}}=\frac{\pi\omega^5}{2g^3\rho}\int_{-\pi}^{\pi}(F_1^2+F_2^2)\,d\psi, \quad .. \quad .. \quad .. \quad (64)$$

where F_1 and F_2 are the real and imaginary parts of

$$\bar{F}\left(\frac{\omega^2}{g}\cos\psi,\ \frac{\omega^2}{g}\sin\psi\right).$$

If

$$F(x,y)=f(r),$$

$$W_{\text{av.}}=\frac{\pi^2\omega^5}{\rho g^3}\left[\bar{f}\left(\frac{\omega^2}{g}\right)\right]^2$$

which is the expression derived by Wehausen and Laitone (1960, 21.9).

Particular cases. Central pressures over elliptic regions

Some of these are defined by

$$p_0(x,y,t)=F(x,y)\begin{matrix}\cos\\ \sin\end{matrix}(\omega t)$$

where

$$\left.\begin{aligned}F(x,y)&=\frac{P\nu}{\pi ab}\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)^{\nu-1}, \quad \text{inside} \quad \frac{x^2}{a^2}+\frac{y^2}{b^2}=1\\ &=0 \qquad\qquad\quad , \quad \text{outside} \quad ,,\end{aligned}\right\} \quad .. \quad (65)$$

For $\nu = 1, \frac{3}{2}, 2$, these disturbances may be characterized as (elliptic) cylindrical,

ellipsoidal and (elliptic) paraboloidal pressures respectively, P being the total pressure on the surface in each case. We have

$$\bar{F}(K_1, K_2) = \frac{2^{\nu-1}P\nu\Gamma(\nu)}{\pi} \cdot \frac{J_\nu\left[(K_1^2a^2+K_2^2b^2)^{\frac{1}{2}}\right]}{(K_1^2a^2+K_2^2b^2)^{\nu/2}} \quad .. \quad .. \quad (66)$$

(Erdélyi 1954, 1.3 (8), 1.13).

(*A*) *Unsteady waves*

For a pressure $F(x, y) \cos \omega t$, we have, from (39),

$$\zeta_1 = \frac{2^{\nu-\frac{1}{2}}P\nu\Gamma(\nu)\omega^2}{\pi\rho ag^2r'(t'^2E)^\nu}\left[J_\nu(t'^2E)\cos(r't'^2) - \frac{1}{2r't'^2}\sin(r't'^2)\right.$$
$$\times\left\{(t'^2E)^2\left(\frac{e^4\sin^2 2\theta}{4E^4} - 2\right)J_{\nu+2}(t'^2E)\right.$$
$$\left.\left.+t'^2E\left(3+\frac{e^2\cos 2\theta}{E^2}\right)J_{\nu+1}(t'^2E)+\tfrac{1}{4}J_\nu(t'^2E)\right\}\right] \quad .. \quad .. \quad (67)$$

under the restrictions (32),
where

$$t' = \frac{(agt^2)^{\frac{1}{2}}}{2r}, \; r' = \frac{r}{a}, \; E = (1-e^2\sin^2\theta)^{\frac{1}{2}}, \quad .. \quad .. \quad (68)$$

e being the eccentricity of the ellipse.
For a pressure $F(x, y) \sin \omega t$, we have, from (39),

$$\zeta_1 = \frac{2^{\nu-\frac{1}{2}}P\nu\Gamma(\nu)\omega t'}{\pi\rho(ag)^{\frac{1}{2}}r'(t'^2E)^\nu}\left[J_\nu(t'^2E)\sin(r't'^2)+\frac{1}{2r't'^2}\cos(r't'^2)\right.$$
$$\times\left\{(t'^2E)^2\left(\frac{e^4\sin^2 2\theta}{4E^4} - 2\right)J_{\nu+2}(t'^2E)\right.$$
$$\left.\left.+t'^2E\left(5+\frac{e^2\cos 2\theta}{E^2}\right)J_{\nu+1}(t'^2E)+\tfrac{1}{4}J_\nu(t'^2E)\right\}\right] \quad .. \quad .. \quad (69)$$

under the restrictions (32).
When $a = b$, we get the corresponding expressions for the symmetrical pressures over circular regions.

(*B*) *Steady waves:*

(i) *Exact solution for symmetrical pressures.*

Let

$$\left.\begin{aligned} p_0(x, y, t) &= \frac{P\nu}{\pi a^2}\left(1-\frac{r^2}{a^2}\right)^{\nu-1}e^{i\omega t}, \; r < a \\ &= 0 \qquad\qquad\qquad\quad , \; r > a \end{aligned}\right\} . \quad .. \quad .. \quad (70)$$

Therefore

$$\bar{f}(K) = \frac{2^{\nu-1}\nu P\Gamma(\nu)}{\pi} \cdot \frac{J_\nu(Ka)}{(Ka)^\nu}$$

Also,

$$\int_0^a \alpha f(\alpha)\,{}_2F_1\left(-n+\tfrac{1}{2},\ -n+\tfrac{1}{2};\ 1, \frac{\alpha^2}{r^2}\right) d\alpha = \frac{P}{2\pi}\cdot {}_2F_1\left(-n+\tfrac{1}{2},\ -n+\tfrac{1}{2};\ \nu+1, \frac{a^2}{r^2}\right)$$

(Erdélyi 1953, 2.4 (2)).

Consequently, the exact solution for the wave elevation in the steady state is given by (55)

$$g\rho\zeta_2 e^{-i\omega t} = -\frac{P}{4}\left(\frac{\omega^2}{g}\right)^2 \sum_{n=0}^{\infty} (-1)^n \left(\frac{\omega^2 r}{2g}\right)^{2n-1} [\Gamma(n+\tfrac{1}{2})]^{-2}$$
$$\times\ {}_2F_1\left(-n+\tfrac{1}{2},\ -n+\tfrac{1}{2};\ \nu+1, \frac{a^2}{r^2}\right) + 2^{\nu-2}a^{-\nu}\nu P\Gamma(\nu)\left(\frac{\omega^2}{g}\right)^{2-\nu}$$
$$\times \left[Y_0\left(\frac{\omega^2 r}{g}\right) + 2iJ_0\left(\frac{\omega^2 r}{g}\right)\right] J_\nu\left(\frac{a\omega^2}{g}\right),\ r > a. \quad \ldots \quad \ldots \quad (71)$$

(ii) *Asymptotic solutions for central pressures on elliptic regions.*

For a pressure $F(x, y)\cos\omega t$, $F(x, y)$ being as defined in (65), we have, from (59),

$$g\rho\zeta_2 = -\frac{2^{\nu-\frac{1}{2}}\nu P\Gamma(\nu)\omega^3}{(\pi a r' g^3)^{\frac{1}{2}}(t_1^2 E)^\nu}\left[J_\nu(Et_1^2)\cos\left(r't_1^2-\omega t+\frac{\pi}{4}\right) + \frac{1}{8r'E^2t_1^2}\sin\left(r't_1^2-\omega t+\frac{\pi}{4}\right)\right.$$
$$\left.\times\left\{(et_1)^4\sin^2 2\theta J_{\nu+2}(Et_1^2) + 4E(et_1)^2\cos 2\theta J_{\nu+1}(Et_1^2) + E^2 J_\nu(Et_1^2)\right\}\right] \ldots \quad (72)$$

under the restrictions (58),

where

$$t_1^2 = \frac{a\omega^2}{g}, \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (73)$$

r', e and E being the same as before.

For a pressure $F(x, y)\sin\omega t$, we get

$$g\rho\zeta_2 = \frac{2^{\nu-\frac{1}{2}}\nu P\Gamma(\nu)\omega^3}{(\pi a r' g^3)^{\frac{1}{2}}(Et_1^2)^\nu}\left[J_\nu(Et_1^2)\sin\left(r't_1^2-\omega t+\frac{\pi}{4}\right) - \frac{1}{8r'E^2t_1^2}\cos\left(r't_1^2-\omega t+\frac{\pi}{4}\right)\right.$$
$$\left.\times\left\{(et_1)^4\sin^2 2\theta J_{\nu+2}(Et_1^2) + 4E(et_1)^2\cos 2\theta J_{\nu+1}(Et_1^2) + E^2 J_\nu(Et_1^2)\right\}\right] \ldots \quad (74)$$

under the restrictions (58).

When $a = b$, we get the corresponding expressions for the symmetrical pressures over circular regions.

Discussions

The similarity of the real and imaginary parts of (39) with the expressions for the elevation of Cauchy-Poisson waves due to an initial surface elevation and impulse suggests that the unsteady waves given by (59) behave, in many respects, like the corresponding Cauchy-Poisson waves for large values of r and

t and for certain frequencies determined by (32). This similarity is further evident in the law of decay of the unsteady state as shown in equation (42).

As to the effect of the frequency, we observe (eqn. (39)) that, in the range of approximations (32), the amplitude of the unsteady waves varies as the frequency or as its square according as the impressed pressure has the phase $\sin \omega t$ or $\cos \omega t$.

At large distances from the seat of the disturbance in the steady state, the third and fourth terms of the expansion (60) for a symmetrical pressure $f(r)\, e^{i\omega t}$ over a circle of radius a are respectively

$$\left.\begin{aligned} &-\frac{9i}{128\omega}\left(\frac{2\pi g}{r^5}\right)^{\frac{1}{2}} \bar{f}\left(\frac{\omega^2}{g}\right) e^{-i\left(\frac{\omega^2 r}{g}-\omega t-\frac{\pi}{4}\right)} \\ \text{and} \\ &-\frac{g}{\omega^2 r^3}\, e^{i\omega t}\int_0^a \alpha f(\alpha)\, d\alpha. \end{aligned}\right\} \quad (75)$$

If, now, ω and a be such that $\bar{f}\left(\frac{\omega^2}{g}\right) = 0$, equations (60) and (75) show that the fourth order term is dominant and this term represents a standing wave of amplitude $O\left(\frac{1}{r^3}\right)$ as $r \to \infty$. Further, equation (59) shows that this phenomenon does not generally occur in motions due to asymmetrical pressure distributions.

Lastly, we note that in any type of pressure distributions, as equation (59) shows, the amplitude of the diverging progressive steady waves of length $\frac{2\pi g}{\omega^2}$ dies out like $r^{-\frac{1}{2}}$ as $r \to \infty$.

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